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RESEARCH REPORT
N° 8171
December 2012
Project-Teams MASCOTTE

ISSN 0249-6399 ISRN INRIA/RR--8171--FR+ENG

Complexity of greedy edge-colouring

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Project-Teams MASCOTTE

Research Report n° 8171 — December 2012 — 10 pages

Abstract: The *Grundy index* of a graph $G = (V, E)$ is the greatest number of colours that the greedy edge-colouring algorithm can use on G. We prove that the problem of determining the Grundy index of a graph $G = (V, E)$ is NP-hard for general graphs. We also show that this problem is polynomial-time solvable for caterpillars. More specifically, we prove that the Grundy index of a caterpillar is $\Delta(G)$ or $\Delta(G) + 1$ and present a polynomial-time algorithm to determine it exactly.

Key-words: Edge colouring, greedy colouring, greedy algorithm, line graphs, caterpillars, NP-complete.

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Complexité de coloration gloutonne d'arêtes

Résumé : L'*indice Grundy* d'un graphe G est le plus grand nombre de couleurs que l'algorithme glouton de coloration d'arêtes peut utiliser pour G. Nous prouvons que le problème de déterminer l'indice Grundy d'un graphe est NP-dur en général. Nous montrons également que ce problème peut être résolu en temps polynômial pour les chenilles. Plus spécifiquement, nous prouvons que l'indice Grundy d'une chenille est $\Delta(G)$ ou $\Delta(G)+1$ et nous présentons un algorithme polynômial pour le déterminer exactement.

Mots-clés : Coloration d'arêtes, coloration gloutonne, algorithme glouton, graphe des lignes, chenilles, NP-complet.

1 Introduction

All the graphs considered in this paper are loopless.

A *k-colouring* of a graph $G = (V, E)$ is a surjective mapping $c : V \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for any edge $uv \in E$. The *chromatic number* is $\chi(G) = min\{k \mid G$ admits a k-colouring$\}$. On the algorithmic point of view, finding the chromatic number of a graph is a hard problem. For all $k \geq 3$ it is NP-complete to decide if a graph admits a k-colouring (see [2]). Furthermore, it is NP-hard to approximate the chromatic number within $|V(G)|^{\varepsilon_0}$ for some positive constant ε_0 as shown by Lund and Yannakakis [5].

Hence lots of heuristics have been developed to colour a graph. The most basic and widespread because it works on-line is the greedy algorithm. Given a vertex ordering $\sigma = v_1 < \cdots < v_n$ of $V(G)$, this algorithm colours the vertices in the order $v_1, \ldots, v_n$, assigning to v_i the smallest positive integer not used on its lower-indexed neighbours. A colouring resulting of the greedy algorithm is called a *greedy colouring*. The *Grundy number* $\Gamma(G)$ is the largest k such that G has a greedy k-colouring. Easily, $\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1$.

Zaker [6] showed that for any fixed k, one can decide in polynomial time if a given graph has Grundy number at most k. However determining the Grundy number of a graph is NP-hard [6], and given a graph G, it is even NP-complete to decide if $\Gamma(G) = \Delta(G)+1$ as shown by Havet and Sampaio [3]. In addition, Asté et al. [1] showed that for any constant $c \geq 1$, it is NP-complete to decide if $\Gamma(G) \leq c \cdot \chi(G)$.

Graph colouring of many graph classes has also been studied. One of the classes is the one of line graphs. The *line graph* of a graph G, denoted $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ whenever e and f share an endvertex. Colouring line graphs corresponds to edge-colouring. A *k-edge-colouring* of a graph G is a surjective mapping $\phi : E(G) \to \{1, \ldots, k\}$ such that if two edges e and f are adjacent (i.e share an endvertex), then $\phi(e) \neq \phi(f)$. A *k-edge colouring* may also be seen as a partition of the edge set of G into k disjoint *matchings* $M_i = \{e \mid \phi(e) = i\}$, $1 \leq i \leq k$. By edge-colouring we mean either the mapping ϕ or the partition.

The *chromatic index* $\chi'(G)$ of a graph G is the least k such that G admits a k-edge-colouring. It is easy to see that $\chi'(G) = \chi(L(G))$. Obviously, $\Delta(G) \leq \chi'(G)$ and Shannon's and Vizing's theorems state that $\chi'(G) \leq \max\{\frac{3}{2}\Delta(G); \Delta(G) + \mu(G)\}$, where $\mu(G)$ is the maximum number of edges between two vertices of G. Holyer [4] showed that for any $k \geq 3$, it is NP-complete to decide if a k-regular graph has chromatic index k.

One can apply the greedy algorithm to colour a line graph. It corresponds to the following greedy algorithm for edge-colouring. Given a graph $G = (V, E)$ and an edge ordering $\theta = e_1 < \cdots < e_n$, assign to e_i the least positive integer that was not already assigned to lower-indexed edges adjacent to it. An edge-colouring obtained by this process is called a *greedy edge-colouring* and it has the following property:

For every $j < i$, every edge e in M_i is adjacent to an edge in M_j. (P)

Note that an edge-colouring satisfying (P) is a greedy edge-colouring relative to any edge ordering in which the edges of M_i precede those of M_j when $i < j$.

The *Grundy index* $\Gamma'(G)$ of a graph G is the largest number of colours of a greedy edge-colouring of G. Notice that $\Gamma'(G) = \Gamma(L(G))$. By definition, $\chi'(G) \leq \Gamma'(G)$. Furthermore, as an edge is incident to at most $2\Delta(G) - 2$ other edges ($\Delta(G) - 1$ at each endvertex), colouring the edges greedily uses at most $2\Delta(G) - 1$ colours. So $\Delta(G) \leq \Gamma'(G) \leq 2\Delta(G) - 1$. There are graphs for which the Grundy index equals the maximum degree: stars for example. On the opposite, for any Δ there is a tree with maximum degree Δ and Grundy index $2\Delta(G) - 1$.

In this paper, we study the complexity of finding the Grundy index of a graph. We prove that it is NP-hard by showing that the following problem is co-NP-complete.

MINIMUM GREEDY EDGE-COLOURING
Instance: A graph G.
Question: $\Gamma'(G) = \Delta(G)$?

The proof is a reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS which was proved to be NP-complete by Holyer [4]. We recall that a *cubic graph* is a 3-regular graph. The reduction also proves that it is co-NP-complete to decide if $\Gamma'(G) = \chi'(G)$.

3-EDGE-COLOURABILITY OF CUBIC GRAPHS
Instance: A cubic graph G.
Question: Is G 3-edge colourable?

We then extend the result to a more general problem.

f-GREEDY EDGE-COLOURING
Instance: A graph G.
Question: $\Gamma'(G) \leq f(\Delta(G))$?

We show that for any function f such that $k \leq f(k) \leq 2k-2$, the problem f-GREEDY EDGE-COLOURING is co-NP-Complete.

Since determining the Grundy index is NP-hard, a natural question to ask is for which class of graphs it can be done in polynomial time. In Section 3, we consider *caterpillars* which are trees such that the deletion of all leaves results in a path, called *backbone*. We show that if T is a caterpillar then $\Gamma'(T) \leq \Delta(T) + 1$ and then give a linear-time algorithm to compute the Grundy index of a caterpillar.

2 Co-NP-completeness results

The aim of this section is to prove that f-GREEDY EDGE-COLOURING is co-NP-complete for every function f such that $k \leq f(k) \leq 2k-2$ for all k.

For sake of clarity, we first show that MINIMUM GREEDY EDGE-COLOURING is co-NP-Complete.

Theorem 1. MINIMUM GREEDY EDGE-COLOURING *is co-NP-Complete.*

MINIMUM GREEDY EDGE-COLOURING is clearly in co-NP, because a greedy edge-colouring of a graph G with at least $\Delta(G)+1$ colours is a certificate that $\Gamma'(G) > \Delta(G)$.

We now prove the co-NP-completeness by reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS.

Let H be a cubic graph on n vertices $w_1, \dots w_n$. Let G be the graph defined by $V(G) = V(H) \cup \{u_1, \dots, u_n\} \cup \{v, a, b, c\}$ and $E(G) = E(H) \cup \{u_i w_i \mid 1 \leq i \leq n\} \cup \{v u_i \mid 1 \leq i \leq n\} \cup \{av, bv, cv\}$. See Figure 1.

In G, $d(v) = n+3$, while the degree of all other vertices is at most 4. Thus, $\Delta(G) = d(v) = n+3$ because $n \geq 4$ has H is cubic. Moreover, every edge of G is adjacent to at most $n+3$ edges so $\Gamma'(G) \leq n+4 = \Delta(G)+1$. Hence the Grundy index of G is either $\Delta(G)$ or $\Delta(G)+1$. The co-NP-completeness of MINIMUM GREEDY EDGE-COLOURING follows directly from the following claim.

Claim 1. $\chi'(H) = 3$ *if and only if* $\Gamma'(G) = \Delta(G) + 1$.

Proof. ($\Rightarrow$) Suppose that there exists a 3-edge-colouring ϕ of H. Let us extend ϕ into a greedy edge-colouring of G with $\Delta(G)+1 = n+4$ colours. Set $\phi(av) = 1$, $\phi(bv) = 2$, $\phi(cv) = 3$, and for all $1 \leq i \leq n$, $\phi(u_i w_i) = 4$ and $\phi(u_i v) = i+4$. Notice that every vertex w_i is incident to an

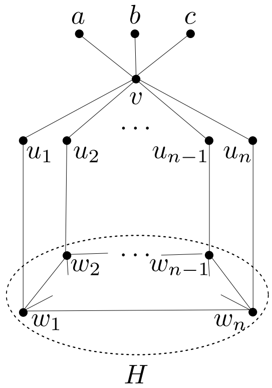

Figure 1: Graph G obtained from a cubic graph H.

edge of H of each colour in $\{1, 2, 3\}$ since H is cubic. Then it is straightforward to check that ϕ is a greedy $(n + 4)$-edge-colouring of G.

($\Leftarrow$) Suppose that there is a greedy $(n+4)$-edge-colouring of G. Some edge is coloured $n+4$. But such an edge has to be adjacent to at least $n + 3$ edges and thus to be one of the vu_i, say vu_n. The edge vu_n is adjacent to exactly $n + 3$ edges. So by Property (P), all edges adjacent to vu_n receive distinct colours in $\{1, \dots, n + 3\}$.

Let us first prove by induction on $1 \leq j \leq n$ that the edge e_j incident to vu_n labelled $n+5-j$ is one of the vu_i, the result holding for $j = 1$. Suppose now that $j \geq 2$. The edge e_j must have degree at least $n+5-j$ since it is adjacent to vu_n and one edge of each colour in $\{1, \dots, n+4-j\}$ by Property (P). Hence e_j must be incident to v since u_nw_n is adjacent to four edges. Then e_j must have degree at least $n + 3$ since it is adjacent to the $j - 1$ edges e_l for $1 \leq l < j$ and one edge of each colour in $\{1, \dots, n + 4 - j\}$. Hence e_j is one of the vu_i.

Hence, without loss of generality, we may assume that $\phi(vu_i) = i + 4$ for all $1 \leq i \leq n$. The edge vu_i is adjacent to an edge coloured 4. This edge must be u_iw_i since the edges av, bv and cv are adjacent to at most 2 edges coloured in $\{1, 2, 3\}$. Thus $\phi(u_iw_i) = 4$ for all $1 \leq i \leq n$.

Now every edge u_iw_i is adjacent to three edges, one of each colour in $\{1, 2, 3\}$. Since $\phi(vu_i) \geq 5$, these three edges must be the three edges incident to w_i in H. Thus all the edges of H are coloured in $\{1, 2, 3\}$. Hence the restriction to ϕ to H is a 3-edge-colouring. $\square$

Remark 1. Observe that the graph G has chromatic index $\Delta(G)$. Indeed colour the edges adjacent to v with the colours $1, \dots, \Delta(G)$ and then extend greedily this colouring to the other edges. Since all the remaining edges are adjacent to at most 6 edges they will all get a colour less or equal to 7. Since $\Delta(G) \geq 7$, we obtain a $\Delta(G)$-edge colouring. Hence the above reduction shows that it is co-NP-complete to decide if $\Gamma'(G) = \chi'(G)$.

Theorem 1 may be generalized as follows.

Theorem 2. *Let f be a function such that $k \leq f(k) \leq 2k - 2$ for all $k \in \mathbb{N}$. f-*GREEDY EDGE-COLOURING *is co-NP-Complete.*

Proof. f-GREEDY EDGE-COLOURING is clearly in co-NP, because a greedy edge-colouring of a graph G with more than $f(\Delta(G))$ colours is a certificate that $\Gamma'(G) > f(\Delta(G))$.

We now prove the co-NP-completeness by reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS.

Let H be a cubic graph on n vertices $w_1, \dots, w_n$ and let G be the graph defined as in the proof of Theorem 1. Set $p = f(n+3) - (n+3)$. Then $0 \leq p \leq n+1$. For $1 \leq i \leq p$, let T_i be the tree with vertex set $\{a_i, b_i, c_i, t_i\} \cup \{a_{i,j}, b_{i,j}, c_{i,j}, s_{i,j}, t_{i,j} \mid 1 \leq j \leq n-1\}$ and edge set $\{a_i t_i, b_i t_i, c_i t_i\} \cup \bigcup_{j=1}^{n-1} \{a_{i,j} t_{i,j}, b_{i,j} t_{i,j}, c_{i,j} t_{ij}, t_{i,j} s_{i,j}, s_{i,j} t_i\}$. Let G' be a graph obtained from the disjoint union of G and the T_i by adding the edge $u_n t_i$ for all $1 \leq i \leq p$. See Figure 2.

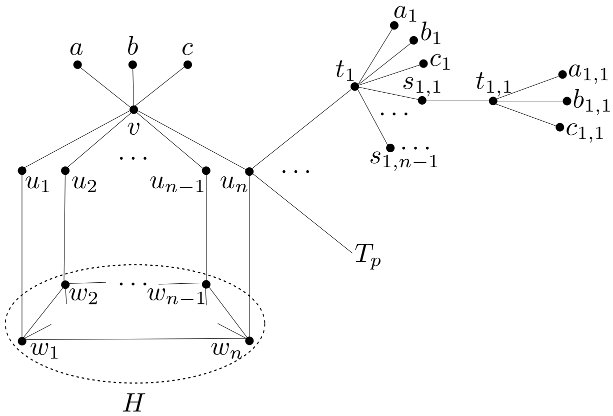

Figure 2: The graph G' obtained from a cubic graph H.

Observe that $\Delta(G') = n+3$ and the vertices of degree $n+3$ are v, $t_1, \dots, t_p$ and u_n when $p = n+1$. Moreover every edge is adjacent to at most $n+3+p$, so $\Gamma'(G) \leq n+3+p+1 = f(\Delta(G')+1$. The co-NP-completeness of f-Greedy Edge-Colouring follows directly from the following claim.

Claim 2. $\chi'(H) = 3$ *if and only if* $\Gamma'(G') = f(\Delta(G')) + 1$.

($\Rightarrow$) Suppose that there exists a 3-edge-colouring ϕ of H. Let us extend ϕ into a greedy edge-colouring of G' with $f(\Delta(G')) + 1 = n + p + 4$ colours. We first extend it into a greedy $(n+4)$-colouring of G as we did in the proof of Theorem 1. In particular, we have $\phi(u_n w_n) = 4$ and $\phi(u_n v) = n+4$. For all $1 \leq i \leq p$ and all $1 \leq j \leq n-1$, we set $\phi(t_i a_i) = 1$, $\phi(t_i b_i) = 2$, $\phi(t_i c_i) = 3$, $\phi(t_{i,j} a_{i,j}) = 1$, $\phi(t_{i,j} b_{i,j}) = 2$, $\phi(t_{i,j} c_{i,j}) = 3$, $\phi(t_{i,j} s_{i,j}) = j+3$, and $\phi(t_i u_n) = n+4+i$. Then it is straightforward to check that ϕ is a greedy $(n+p+4)$-edge-colouring of G'.

($\Leftarrow$) Suppose that G' admits a greedy $(n+p+4)$-edge-colouring ϕ. For all $1 \leq i \leq p$, there is an edge e_i coloured $n+4+i$. This edge must has to be adjacent to at least $n+3+i$ edges by Property (P). So all the e_i must be in $F = \{vu_n\} \cup \{u_n t_i \mid 1 \leq i \leq p\}$. Now the edge e_p is adjacent to an edge e_0 coloured $n+4$. This edge is adjacent to at least $n+4$-edges, one of each colour in $\{1, \dots, n+3\}$ and e_p. Hence it also has to be in F. Since $|F|$, all the edges in F are coloured with distinct labels in $\{n+4, \dots, n+p+4\}$.

Now applying the same reasoning as in the proof of Theorem 1, we derive that the restriction to ϕ to H is a 3-edge-colouring. □

3 Greedy edge-colouring of caterpillars

In this section, we show a polynomial-time algorithm solving Greedy Edge-Colouring for caterpillars. Recall that caterpillars are trees such that the deletion of all leaves results in a path, called backbone.

We first show some properties of a greedy edge-colouring of a caterpillar where we can see that the Grundy index of a caterpillar T is at most $\Delta(T)+1$. We then give a polynomial-time algorithm that computes its Grundy index.

3.1 Grundy index of a caterpillar

Lemma 3. *Let T be a caterpillar and v a vertex in its backbone. In every greedy edge-colouring of T, the colours $1,\ldots,d(v)-2$ appear on the edges incident to v.*

Proof. By the contrapositive. Let c be a greedy edge-colouring of T. Suppose that a colour $\alpha\in\{1,\ldots,d(v)-2\}$ is not assigned to any edge incident to v. Then, since all the edges incident to v have different colours, at least three colours strictly greater than $d(v)-2$ appear on three edges incidents to v. One of these colours, say β must appear on an edge e incident with a leaf. But e is uniquely adjacent to edges incident to v. So e is adjacent to no edge coloured α. Since $\alpha\le d(v)-2<\beta$, the edge-colouring c is not greedy. □

Lemma 4. *Let c be a greedy edge-colouring of a caterpillar T and v a vertex in the backbone of T. If two edges e_1 and e_2 incident to v receive colours greater than $d(v)-1$, then e_1 and e_2 are two edges of the backbone and the edges incident to v and leaves are coloured $1,\ldots,d(v)-2$.*

Proof. Suppose by way of contradiction that one of these two edges, say e_1, is incident to a leaf. Then e_1 is adjacent to $d(v)-1$ other edges, and one of them, namely e_2, is assigned a colour greater than $d(v)-1$. Thus e_1 is adjacent to at most $d(v)-2$ edges whose colour is less or equal to $d(v)-1$. So, there is a colour α in $\{1,\ldots,d(v)-1\}$ such that no edge incident to e_1 is coloured α. This contradicts the fact that c is greedy. Hence e_1 and e_2 are edges of the backbone.

Now by Lemma 3, there must be edges incident to v of each colour in $\{1,\ldots,d(v)-2\}$. So the $d(v)-2$ edges distinct form e_1 and e_2, which are the edges linking v and leaves are coloured in $\{1,\ldots,d(v)-2\}$. □

We use Lemma 4 to prove the following theorem.

Theorem 5. *If T is a caterpillar, then $\Gamma'(T)\le\Delta(T)+1$.*

Proof. Set $\Delta(T)=\Delta$. Suppose by way of contradiction that it is possible to greedily colour T with $\Delta+2$ colours. Let e be an edge coloured $\Delta+2$. It must be adjacent to at least $\Delta+1$ edges, one of each colour $1,\ldots,\Delta+1$. Thus, the edge e is in the backbone. According to Lemma 4, the edges e_1 and e_2 adjacent to e with colours Δ and $\Delta+1$ are in the backbone. Furthermore all the edges adjacent to e which are neither e_1 nor e_2 are coloured in $\{1,\ldots,\Delta-2\}$. Hence e is adjacent to no edge coloured $\Delta-1$, a contradiction. □

Theorem 5 is tight. For instance, consider the caterpillar T_k with backbone (t,u,v,w) for which the vertex t has degree $k-1$, u has degree 2 and v and w degree k. An edge-colouring in which the $k-2$ edges incident to t and a leaf are coloured with $1,\ldots,k-2$, the $k-1$ edges incident to w and a leaf with $1,\ldots,k-1$, the $k-2$ edges incident to v and a leaf with $1,\ldots,k-2$, the edge tu with $k-1$, the edge vw with k and the edge uv with $k+1$ is greedy.

3.2 Finding the Grundy index of a caterpillar

Let us now examine in more details when a caterpillar T has Grundy index $\Delta(T)+1$.

Let T be a caterpillar with backbone $P=(v_1,v_2,\ldots,v_n)$. The *first edge* of P is v_1v_2. For any edge $e=v_iv_{i+1}\in P$, removing e from T gives two caterpillars T_e^- and T_e^+, the first one containing v_i and the second one containing v_{i+1}. For convenience, the backbone of T_e^- is

$P_e^- = (v_i, v_{i-1}, \dots, v_1)$ and the backbone of T_e^+ is $P_e^+ = (v_{i+1}, \dots, v_n)$. Hence the first edge of T_e^- is (v_i, v_{i-1}) and the first edge of T_e^+ is (v_{i+1}, v_{i+2}).

Lemma 6. *Let T be a caterpillar of maximum degree Δ with backbone $P = (v_1, \dots, v_n)$. Then $\Gamma'(T) = \Delta + 1$ if and only if there is an edge $e \in E(P) \setminus \{v_1v_2, v_{n-1}v_n\}$ such that*

1. *one endvertex of e has degree Δ, and*
2. *one of the two caterpillars T_e^- and T_e^+ has a greedy edge-colouring such that the first edge of its backbone is coloured Δ and the other has a greedy edge-colouring such that its first edge of its backbone is coloured $\Delta - 1$.*

Proof. Assume that T has a greedy $(\Delta+1)$-edge-colouring. Let e be an edge coloured $\Delta+1$. By Lemma 4, e is in the backbone and incident to a vertex of degree Δ, proving (i). Moreover, the edge e is adjacent to an edge coloured Δ and another one labelled $\Delta-1$. Again by Lemma 4 these two edges must also be in the backbone. In particular, e is not v_1v_2 nor $v_{n-1}v_n$ because these two edges are adjacent to a unique edge of the backbone. Moreover the greedy edge-colourings induced on T_e^- and T_e^+ clearly satisfy (ii).

Conversely, assume that there is an edge $e \in E(P) \setminus \{v_1v_2, v_{n-1}v_n\}$ satisfying (i) and (ii). Let ϕ^- and ϕ^+ be the greedy edge-colourings of T_e^- and T_e^+ respectively as in (ii). Let ϕ be the edge-colouring of T defined by $\phi(e) = \Delta+1$, $\phi(f) = \phi^-(f)$ for all $f \in T_e^-$ and $\phi(f) = \phi^+(f)$ for all $f \in T_e^+$. We claim that ϕ is a greedy edge-colouring. Clearly, since ϕ^- and ϕ^+ are greedy, it suffices to prove that e is adjacent to an edge of every colour i in $\{1, \dots, \Delta\}$. Since ϕ^+ and ϕ^- satisfy (ii), then e is adjacent to an edge labelled Δ and an edge labelled $\Delta-1$. Now, e is incident to a vertex v of degree Δ. This vertex is incident to e and an edge f in the backbone. The edge f is the first edge of a tree T_f in $\{T_e^+, T_e^-\}$. In the greedy edge-colouring of T_f, the edge f has a colour greater than $\Delta - 2$, so the $\Delta - 2$ edges incident to v which are not e nor f have all one colour in $1, \dots, \Delta - 2$. Hence e is adjacent to an edge of every colour in $\{1, \dots, \Delta\}$. □

In view of this lemma, it is useful to decide when a caterpillar has a greedy edge-colouring with a prescribed colour on the first edge of its backbone.

Lemma 7. *Let T be a caterpillar with backbone P with first edge is $e = uv$. Then T has a greedy edge-colouring such that e is coloured k if and only if one of the following holds:*

1. *$d(u) \geq k$ or $d(v) \geq k$;*
2. *$d(u) = k - 1$ and T_e^+ admits a greedy edge-colouring such that the first edge of P_e^+ is coloured $k - 1$.*

Proof. Let $e = uv$ with u the first vertex of P. Assume first that T has a greedy edge-colouring such that e is coloured k and that e is incident to no vertex of degree k. Then the edges incident to u must be coloured by $1, \dots, d(u) - 1$ and the edges incident to v and a leaf are coloured by $1, \dots d(v) - 2$. Hence the edge incident to e and coloured $k - 1$ must be the first edge of P_e^+ is coloured $k - 1$ by Property (P). So the edge incident to e and coloured $k - 2$ must be incident to u, and thus $d(u) - 1 \geq k - 2$, that is $d(u) \geq k - 1$.

Assume now that (i) holds. Let x be a vertex in $\{u, v\}$ with degree at least k. One can colour all the edges incident to x with $1, \dots, d(v)$ such that e is coloured k and then extend this edge-colouring greedily to obtain the desired greedy edge-colouring of T.

Finally assume that (ii) holds. Let ϕ be a greedy-edge colouring of T_e^+ such that the first edge of P_e^+ is coloured $k-1$. One can extend it by assigning k to e, $1, \dots, k-2$ to the $k-2$ edges incident to u and leaves . It is routine to check that this is a greedy-edge colouring of T. □

Lemma 7 translates easily in a linear-time recursive algorithm. Using this, and Lemma 6, one derive a linear-time algorithm for computing the Grundy index of a given caterpillar.

Theorem 8. *Determining the Grundy index of a caterpillar T can be done in $O(|V(T)|)$.*

Proof. Theorem 5 and Lemma 6 imply that Algorithm 1 return the Grundy index of T provided that we have a subroutine FirstEdge(T,P,k) that returns 'yes' if a caterpillar T with backbone P admits a greedy-edge colouring such that the first edge of P is coloured k.

Algorithm 1: GrundyIndex(T)
Input: A caterpillar T.
Output: $\Gamma'(T)$.

(1) Let $P = (v_1, v_2, ..., v_n)$ be the backbone of T. Compute $d(v_i)$ for all $i \leq v_n$ and compute $\Delta = \Delta(T)$.

(2) **for** $i = 2$ to $n - 2$ **do**

  (3) $e := v_i v_{i+1}$;

  (4) **if** $d(v_i) = \Delta$ or $d(v_{i+1}) = \Delta$ **then**

    (5) **if** FirstEdge(T_e^+, P_e^+, Δ)=TRUE and FirstEdge(T_e^-, P_e^-, $\Delta - 1$)=TRUE **then**

    (6) return $\Delta + 1$;

    (7) **if** FirstEdge(T_e^+ , P_e^+ , $\Delta-1$)=TRUE and FirstEdge(T_e^- , T_e^- , Δ)=TRUE **then**

    (8) return Δ+1;

(9) Return Δ;

Such a subroutine FirstEdge may be obtained by Algorithm 2 according to Lemma 7.

Algorithm 2: FirstEdge
Input: A caterpillar T with backbone P and an integer k.
Output: $TRUE$ if there is a greedy k-edge-colouring of T with first edge of P coloured k, and $FALSE$ otherwise.

(1) Let u be the first vertex of P and v its second. (So uv is the first edge.)

(2) **If** $d(u) \geq k$ or $d(v) \geq k$ **then**

  (3) return $TRUE$;

(4) **If** $d(u) \geq k - 1$ **then**

  (5) return FirstEdge($T - u$, $P - u$, $k - 1$);

(6) return $FALSE$;

Let us now examine the complexity of Algorithm 1. Let us first observe that FirstEdge(T, P, k) makes a constant number of operations before calling FirstEdge($T - u$, $P - u$, $k - 1$). Hence an easy induction show that it makes $O(k)$ operations in total.

Algorithm 1 first compute (line 1) the degrees of all the v_i, which can be done in time $O(|V(T)|)$ and then takes the maximum of all this values which can also be done in time $O(|V(T)|$.

In a second phase (line 2 to 8), for each edge $e \in P$ which is incident to a vertex of degree Δ, Algorithm 1 makes at most four calls of FirstEdge with last parameter $\Delta - 1$ or Δ. Hence for each $e \in P$ it makes $O(\Delta)$ operations. Let S be the set of vertices of degree Δ, The number of edges of P incident to a vertex of degree Δ is at most $2|S|$. But every vertex in S is adjacent to at least $\Delta - 2$ leaves. Hence $|V(T)| \geq |S| + (\Delta - 2)|S|$, so $|S| \leq |V(T)|/(\Delta - 1)$. Hence, in this second phase, the algorithm makes at most $O\left(2 \times \frac{|V(T)|}{\Delta-1}\Delta\right) = O(|V(T)|)$ operations.

Thus, in total, Algorithm 1 makes $O(|V(T)|)$ operations.

□

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ISSN 0249-6399